Elgersburg Lectures – March 2010

Lecture II

LINEAR TIME-INVARIANT

DIFFERENTIAL SYSTEMS

Theme

Linear time-invariant differential systems are important for several reasons.

- **They occur often in practice (both in technology and as pedagogical examples).**
- **They describe nonlinear systems ‘locally’, through linearization.**
- **They motivate elegant mathematics.**

In this lecture we examine the mathematical structure of LTIDSs.

Outline

- Polynomial matrices and differential operators
- The structure of kernel representations
- Inputs and outputs, the transfer function
- Autonomous systems
- Controllability and image representations
- Rational symbols

Polynomial matrices

LTIDSs

The study of LTIDSs basically deals with real polynomial matrices and linear constant coefficient differential operators.

We therefore first discuss the structure of the set of polynomial vectors and matrices, and linear differential operators with constant coefficients.

Rings

A *ring* is a set R equipped with two binary operations,

$$+ : R \times R \to R \qquad * : R \times R \to R,$$

called *addition* and *multiplication*. Multiplication is usually written by juxtaposition of the multiplied elements, rather that with the $*$ $a * b \to ab$. These operations satisfy:

- **$(R, +)$ is an abelian group with identity element 0,**
- **multiplication is associative, with identity element 1,**
- **multiplication distributes over addition.**

So for all $a, b, c \in R$, there holds $(ab)c = a(bc)$, written as abc, $a1 = 1a = a$, $a(b+c) = ab + ac, (a+b)c = ac + bc$.

Multiplication need not be commutative. If it is, we call the ring a *commutative ring*.

Examples

- **Of commutative rings:**

$$\mathbb{Z}, \mathbb{R}[\xi], \mathbb{R}[\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}}], \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}), \mathbb{R}(\xi).$$

- **Of non-commutative rings:**

$$\mathbb{R}^{\mathtt{n}\times\mathtt{n}}, \mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{n}}, \mathbb{R}[\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}}]^{\mathtt{n}'\times\mathtt{n}'}, \mathbb{R}(\xi)^{\mathtt{n}\times\mathtt{n}}.$$

Units

Every element r of a ring R has a additive inverse, $-r$. But it need not have a multiplicative inverse. For example, in $\mathbb{R}[\xi]$ only the non-zero polynomials of degree 0 have a multiplicative inverse.

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An elements $r \in R$ **is called a unit if it has a multiplicative inverse: if there exists** $r' \in \mathbb{R}$ **such that** $rr' = r'r = 1$**;**
r' **is uniquely determined by** r**, and denoted by** r^{-1}**.**

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The term unimodular is used as a synonym for unit for square polynomial matrices. A unimodular matrix $U \in \mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{n}}$ has therefore an inverse $U^{-1} \in \mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{n}}$ that is also a polynomial matrix. $M \in \mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{n}}$ is unimodular if and only if $\mathtt{determinant}(M)$ is a non-zero polynomial of degree zero, i.e. if and only if $\mathtt{determinant}(M)$ is a unit in $\mathbb{R}[\xi]$.

Modules

Let R be a commutative ring. A module $\mathscr{M}$ over R (also called an R-module) is an abelian group $(\mathscr{M},+)$ with an operation, called *scalar multiplication*, mapping $R \times \mathscr{M} \to \mathscr{M}$. Scalar multiplication is usually written by juxtaposition, i.e., $rm \in \mathscr{M}$ for $r \in R$ and $m \in \mathscr{M}$.

These operations satisfy, for all $r,s \in \mathbb{R}$ and $x,y \in \mathscr{M}$,

- $r(x+y) = rx+ry$,
- $(r+s)x = rx+ry$,
- $(rs)x = r(sx)$, **therefore written as** rsx**,**
- $1x = x$.

In slang, we think of a module as a 'vector space over a ring'.

The following example is especially important to us:
$\mathbb{R}[\xi]^{\mathtt{n}}$ **is a module over** $\mathbb{R}[\xi]$**. So is, of course,** $\mathbb{R}[\xi]^{1\times\mathtt{n}}$**.**

Free modules

An R**-module** $\mathscr{M}$ **is said to be** ***finitely generated*** **if there exist elements** $g_1, g_2, \ldots, g_{\mathtt{r}} \in \mathscr{M}$ **(called** ***generators*** **of** $\mathscr{M}$**), such that each element of** $m \in \mathscr{M}$ **can be written as**

$$m = c_1 g_1 + c_2 g_2 + \cdots + c_{\mathtt{r}} g_{\mathtt{r}}, \ \textbf{with } c_1, c_2, \ldots, c_{\mathtt{r}} \in R.$$

An R**-module** $\mathscr{M}$ **is said to be** ***free*** **if there exist a set of generators** $\{e_1, e_2, \ldots, e_{\mathtt{r}}\}$ **of** $\mathscr{M}$ **(called** ***basis*** **of** $\mathscr{M}$**) such that**

$$c_1 e_1 + c_2 e_2 + \cdots + c_{\mathtt{r}} e_{\mathtt{r}} = 0 \ \textbf{ implies } \ e_1, e_2, \ldots, e_{\mathtt{r}} = 0.$$

The cardinality of the basis is uniquely defined by $\mathscr{M}$**, and is called the** ***dimension*****,** ***rank*****, or** ***order*** **of** $\mathscr{M}$**.**

Submodules of $\mathbb{R}[\xi]^n$

As already mentioned, we are especially interested in the $\mathbb{R}[\xi]$-module $\mathbb{R}[\xi]^{1\times \mathtt{n}}$ and its submodules. These are very tame modules: they are free, hence have a basis, and behave very much like vector spaces.

Let $\mathscr{M}$ be an $\mathbb{R}[\xi]$-submodule of $\mathbb{R}[\xi]^{\mathtt{n}}$. It has a basis, say $\{e_1, e_2, \ldots, e_{\mathtt{r}}\}$. Any other basis $\{e'_1, e'_2, \ldots, e'_{\mathtt{r}}\}$ of $\mathscr{M}$ is obtained by

$$\begin{bmatrix} e'_1 & e'_2 & \cdots & e'_{\mathtt{r}} \end{bmatrix} = U \begin{bmatrix} e_1 & e_2 & \cdots & e_{\mathtt{r}} \end{bmatrix},$$

with $U \in \mathbb{R}[\xi]^{\mathtt{r}\times\mathtt{r}}$ unimodular.

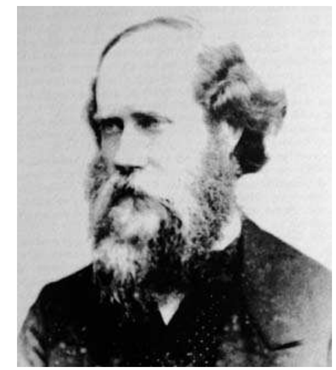
The Smith form

The elements of $\mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$ can be brought into a simple form by pre- and post-multiplication by unimodular matrices.

This canonical form, called the Smith form, comes in very handy in proofs.

<u>Theorem:</u> Let $M \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_1}$. Then there exist $U \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_1}$ and $V \in \mathbb{R}[\xi]^{\mathtt{n}_2 \times \mathtt{n}_2}$, both unimodular, such that

$$UMV = \begin{bmatrix} \mathbf{diag}(d_1, d_2, \ldots, d_{\mathtt{r}}) & 0_{\mathtt{r} \times (\mathtt{n}_2 - \mathtt{r})} \\ 0_{(\mathtt{n}_1 - \mathtt{r}) \times \mathtt{r}} & 0_{(\mathtt{n}_1 - \mathtt{r}) \times (\mathtt{n}_2 - \mathtt{r})} \end{bmatrix},$$



Henry Smith
1826–1883

with $d_1, d_2, \ldots, d_{\mathtt{r}} \in \mathbb{R}[\xi]$ monic, and $d_{\mathtt{k}+1}$ a factor of $d_{\mathtt{k}}$ for $\mathtt{k} = 1, 2, \ldots, \mathtt{r} - 1$. The number $\mathtt{r}$ (the rank of M) and the polynomials $d_1, d_2, \ldots, d_{\mathtt{r}}$ (the invariant polynomials of M) are uniquely defined by M.

The proof is surprisingly simple (see Exercise II.1).

Differential operators

Scalar differential equations

Consider the scalar constant-coefficient linear ODE

$$p_0 w + p_1 \frac{d}{dt} w + \cdots + p_{\mathtt{n}-1} \frac{d^{\mathtt{n}-1}}{dt^{\mathtt{n}-1}} w + p_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w = 0,$$

with $p_0, p_1, \ldots, p_{\mathtt{n}-1}, p_{\mathtt{n}} \in \mathbb{C}$ **(even though we are mainly interested in the real case, it is convenient – notationwise – to consider the complex case).**

In shorthand, with $p(\xi) = p_0 + p_1 \xi + \cdots + p_{\mathtt{n}-1} \xi^{\mathtt{n}-1} + p_{\mathtt{n}} \xi^{\mathtt{n}}$**,**

$$p\left(\frac{d}{dt}\right) w = 0, \qquad p \in \mathbb{C}[\xi]. \qquad (\clubsuit)$$

Question: **which functions** $w : \mathbb{R} \to \mathbb{C}$ **are solutions of** $(\clubsuit)$**?**

Scalar differential equations

$$p\left(\frac{d}{dt}\right)w = 0, \qquad p \in \mathbb{C}[\xi]. \qquad (\clubsuit)$$

The set of solutions can be described very explicitly.

Proposition 1:

Let $\lambda_1, \lambda_2, \ldots, \lambda_{\mathtt{r}} \in \mathbb{C}$ **be the** ***distinct*** **roots of** p **and** $\mathtt{m}_1, \mathtt{m}_2, \ldots, \mathtt{m}_{\mathtt{r}}$ **their multiplicities. Of course,** $\mathtt{m}_1 + \mathtt{m}_2 + \cdots + \mathtt{m}_{\mathtt{r}} = \mathbf{degree}(p)$**.**
$y : \mathbb{R} \to \mathbb{C}$ **is a solution of (♣) if and only it is of the form**

$$y(t) = \pi_1(t)e^{\lambda_1 t} + \pi_2(t)e^{\lambda_2 t} + \cdots + \pi_{\mathtt{r}}(t)e^{\lambda_{\mathtt{r}} t},$$

with $\pi_1, \pi_2, \ldots, \pi_{\mathtt{r}} \in \mathbb{C}[\xi]$ **polynomials of** $\mathbf{degree}(\pi_{\mathtt{k}}) < \mathtt{m}_{\mathtt{k}}$ **for** $\mathtt{k} = 1, 2, \ldots, \mathtt{r}$**.**

For $p \in \mathbb{R}[\xi]$**, and** $y : \mathbb{R} \to \mathbb{R}$**, simply take the real part.**

Proposition 2:

Let $0 \neq p \in \mathbb{R}[\xi]$**, and** $f \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$**. Then there exists** $y \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ **such that**

$$p\left(\frac{d}{dt}\right)y = f.$$

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Propositions 1 and 2 are classical results. Proposition 1 is perhaps the most basic result from the theory of differential equations. Proposition 2 can be refined since the solution space of $p\left(\frac{d}{dt}\right)y = f$ **forms a linear variety of** $\texttt{degree}(p)$**, with one solution for each initial condition**

$$y(0), \frac{d}{dt}y(0), \ldots, \frac{d^{\texttt{degree}(p)-1}}{dt^{\texttt{degree}(p)-1}}y(0).$$

Multivariable differential equations

Proposition 1 can be generalized to multivariable ODEs.
Let $P \in \mathbb{C}[\xi]^{\mathtt{k}\times\mathtt{k}}$, $P(\xi) = P_0 + P_1\xi + \cdots + P_{\mathtt{n}}\xi^{\mathtt{n}}$**, have** $\mathtt{determinant}(P) \neq 0$**. The resulting ODE is**

$$P_0 y + P_1 \frac{d}{dt} y + \cdots + P_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} y = 0, \qquad \textbf{i.e.,}\ P\left(\frac{d}{dt}\right) w = 0. \qquad (\clubsuit\clubsuit)$$

Multivariable differential equations

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The set of solutions $y : \mathbb{R} \to \mathbb{C}^{\mathtt{k}}$ of this ODE can be described as follows.

<u>Proposition 3</u>: Let $\lambda_1, \lambda_2, \ldots, \lambda_{\mathtt{r}} \in \mathbb{C}$ be the *distinct* roots of $\mathtt{determinant}(P)$ and $\mathtt{m}_1, \mathtt{m}_2, \ldots, \mathtt{m}_{\mathtt{r}}$ their multiplicities. The solutions $y : \mathbb{R} \to \mathbb{C}^{\mathtt{k}}$ of ($\clubsuit\clubsuit$) are of the form

$$y(t) = \pi_1(t) e^{\lambda_1 t} + \pi_2(t) e^{\lambda_2 t} + \cdots + \pi_{\mathtt{r}}(t) e^{\lambda_{\mathtt{r}} t},$$

with $\pi_1, \pi_2, \ldots, \pi_{\mathtt{r}} \in \mathbb{C}[\xi]^{\mathtt{k}}$ polynomial vectors. The polynomial vectors $\pi_{\mathtt{k}}$ vary over an $\mathtt{m}_{\mathtt{k}}$-dimensional subspace of $\mathscr{V}_{\mathtt{k}} \subset \mathbb{C}[\xi]^{\mathtt{w}}$ and have $\mathtt{degree}(\pi_{\mathtt{k}}) < \mathtt{m}_{\mathtt{k}}$ for $\mathtt{k} = 1, 2, \ldots, \mathtt{r}$.

The subspaces $\mathscr{V}_{\mathtt{k}}$ can be described precisely in terms of P. We do not enter into these details.

Injectivity, surjectivity, and bijectivity of differential operators

Let $P \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$**, and consider the map**

$$P\left(\frac{d}{dt}\right) : \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}_2}) \to \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}_1}).$$

We study when this map is injective, surjective, or bijective.

Injectivity, surjectivity, and bijectivity of differential operators

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Proposition 4: Let $P \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$. The map $P\left(\frac{d}{dt}\right)$ is

- **injective if and only if the complex matrix $P(\lambda) \in \mathbb{C}^{\mathtt{n}_1 \times \mathtt{n}_2}$ has rank $\mathtt{n}_2$ for all $\lambda \in \mathbb{C}$.**
- **surjective if and only if the polynomial matrix P has rank $\mathtt{n}_1$.**
- **bijective if and only if $\mathtt{n}_1 = \mathtt{n}_2$ and P is unimodular.**

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- **surjective** if and only if the polynomial matrix P has rank $\mathtt{n}_1$.
- **bijective** if and only if $\mathtt{n}_1 = \mathtt{n}_2$ and P is unimodular.

Proof: In the scalar case $\mathtt{n}_1 = \mathtt{n}_2 = 1$, this proposition is a direct consequence of Propositions 1 and 2. Combining the scalar case with the Smith form $\rightsquigarrow$ Proposition 4.

The structure of kernel representations

Kernel representation

$$[\![\mathscr{B} \in \mathscr{L}^{\mathtt{w}}]\!] :\Leftrightarrow [\![\mathscr{B} = \mathtt{kernel}\left(R\left(\frac{d}{dt}\right)\right) \textbf{ for some } R \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}]\!].$$

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This leads to the following question

When do

$$R_1\left(\frac{d}{dt}\right)w = 0 \quad \textit{and} \quad R_2\left(\frac{d}{dt}\right)w = 0$$

determine the same system?

The annihilators

Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$**. The polynomial vector** $n \in \mathbb{R}[\xi]^{1\times\mathtt{w}}$ **is said to be an** $[\![$ **annihilator of** $\mathscr{B}]\!]$ $:\Leftrightarrow [\![n\left(\frac{d}{dt}\right)\mathscr{B} = 0]\!]$**, i.e.,** $n\left(\frac{d}{dt}\right)w = 0$ **for all** $w \in \mathscr{B}$**.**

Denote the set of annihilators of $\mathscr{B}$ **by** $\mathscr{N}_{\mathscr{B}}$**. It is easy to see that** $\mathscr{N}_{\mathscr{B}}$ **is an** $\mathbb{R}[\xi]$**-module. That is,** $n_1, n_2 \in \mathscr{N}_{\mathscr{B}}$ **and** $p \in \mathbb{R}[\xi]$ **imply** $n_1 + pn_2 \in \mathscr{N}_{\mathscr{B}}$**.**

Notation

For $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$**, let** $\langle R \rangle$ **denote the** $\mathbb{R}[\xi]$**-submodule of** $\mathbb{R}[\xi]^{1 \times \mathtt{w}}$ **generated by the rows of** R**.**

Let $\mathscr{M}^{\mathtt{w}}$ **denote the set of** $\mathbb{R}[\xi]$**-submodules of** $\mathbb{R}[\xi]^{1 \times \mathtt{n}}$**.**

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For $M \in \mathscr{M}^{\mathtt{w}}$**, let** $\mathscr{S}_M$ **denote the behavior**

$$\mathscr{S}_M := \{ w \in \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid m\left(\frac{d}{dt}\right) w = 0 \ \textbf{for all} \ m \in M \}.$$

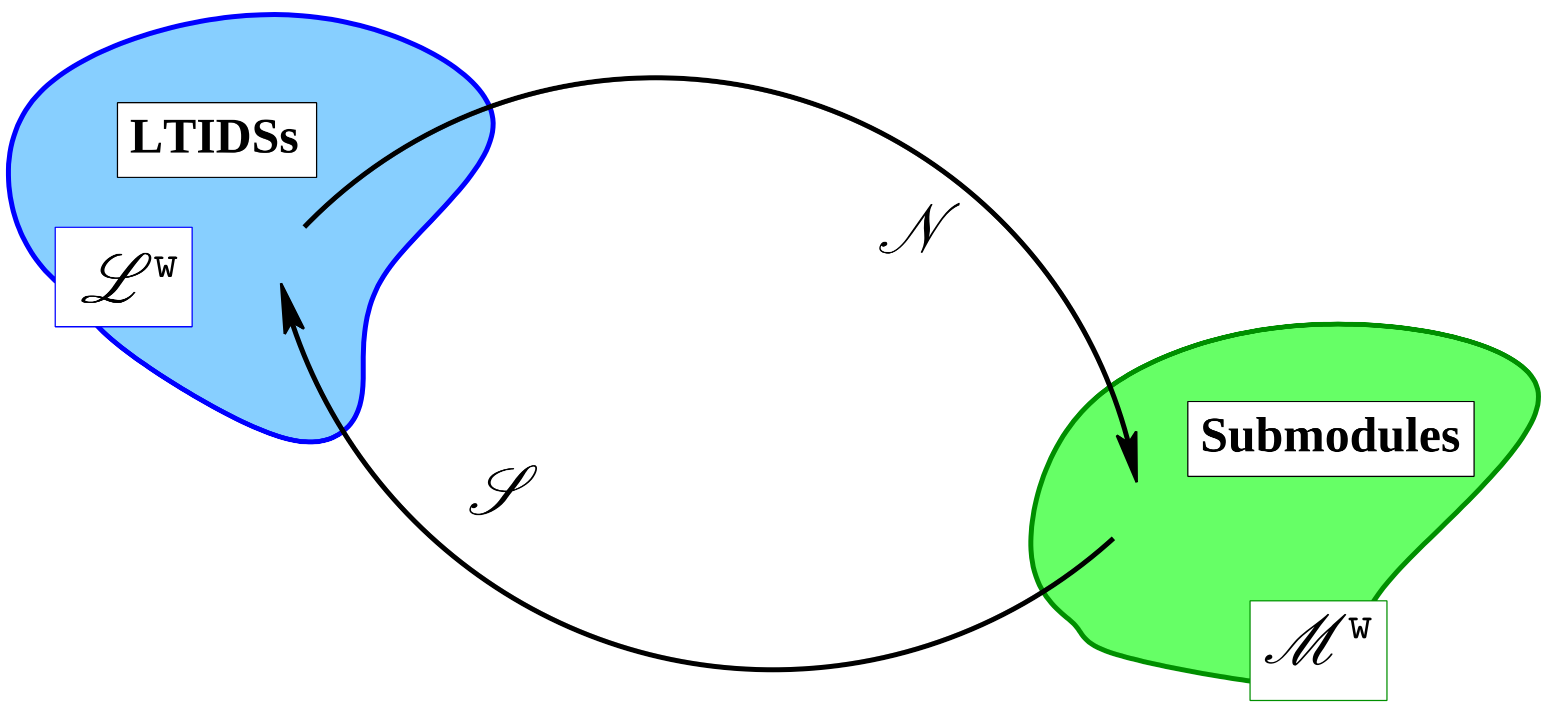
It is easy to see that this behavior belongs to $\mathscr{L}^{\mathtt{w}}$**. In fact, if** $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$ **is a polynomial matrix whose rows are generators of** M**,** $M = \langle R \rangle$**, then** $\mathscr{S}_M = \mathtt{kernel}\left(R\left(\frac{d}{dt}\right)\right)$**.**

From behaviors to $\mathbb{R}[\xi]$-modules and back

Summarizing,

$$\mathscr{N} : \mathscr{L}^{\mathtt{w}} \to \mathscr{M}^{\mathtt{w}}, \qquad \mathscr{B} \overset{\mathscr{N}}{\mapsto} \mathscr{N}_{\mathscr{B}}.$$

$$\mathscr{S} : \mathscr{M}^{\mathtt{w}} \to \mathscr{L}^{\mathtt{w}}, \qquad \mathscr{M}^{\mathtt{w}} \overset{\mathscr{S}}{\mapsto} \mathscr{S}_{M}.$$



The structure theorem

Theorem

1. Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$. Then

$$\left[\!\left[\mathscr{B} = \mathtt{kernel}\left(R\left(\tfrac{d}{dt}\right)\right)\right]\!\right] \Leftrightarrow \left[\!\left[\mathscr{N}_{\mathscr{B}} = \langle R \rangle\right]\!\right].$$

2. Let $\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{L}^{\mathtt{w}}$. Then

$$\left[\!\left[\mathscr{B}_1 = \mathscr{B}_2\right]\!\right] \Leftrightarrow \left[\!\left[\mathscr{N}_{\mathscr{B}_1} = \mathscr{N}_{\mathscr{B}_2}\right]\!\right].$$

3. The maps $\mathscr{N}$ and $\mathscr{S}$ are each other's inverse, i.e.,

$$\mathscr{S}_{\mathscr{N}_{\mathscr{B}}} = \mathscr{B} \quad \text{and} \quad \mathscr{N}_{\mathscr{S}_M} = M.$$

Hence there exists a one-to-one relation between $\mathscr{L}^{\mathtt{w}}$ and the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{\mathtt{w}}$.

Proof in telegram-style

1. **The claim is equivalent to** $\mathscr{N}_{\texttt{kernel}(R(\frac{d}{dt}))} = \langle R \rangle$.

- **First prove the case $\mathtt{w} = 1$ by applying Proposition 1 of the section on differential operators.**
- **Then show that, without loss of generality, it can be assumed that R is in Smith form.**
- **Finally, prove the case that R is in Smith form by repeated application of the case $\mathtt{w} = 1$.**

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3. **is a consequence of 1.**

Example

Let $\mathtt{w} = 1$**. Let** $\mathscr{B}$ **be described by**

$$r_1\left(\frac{d}{dt}\right)w = 0, r_2\left(\frac{d}{dt}\right)w = 0, \ldots, r_{\mathtt{n}}\left(\frac{d}{dt}\right)w = 0,$$

with $r_1, r_2, \ldots, r_{\mathtt{n}} \in \mathbb{R}[\xi]$**. The annihilators consist of all polynomials that have** $r \in \mathbb{R}[\xi]$**, the greatest common divisor of** $r_1, r_2, \ldots, r_{\mathtt{n}}$**, as a factor. Hence**

$$r\left(\frac{d}{dt}\right)w = 0$$

is also a kernel representation of $\mathscr{B}$**.**

The systems $\mathscr{L}^1$ **and the** $\mathbb{R}[\xi]$**-submodules of** $\mathbb{R}[\xi]$ **are hence in** $1 \leftrightarrow 1$ **relation with the monic polynomials in** $\mathbb{R}[\xi]$**.**

Example

Let $\mathtt{w} = 1$. Assume that, instead of taking the $\mathscr{C}^\infty(\mathbb{R},\mathbb{R})$-solutions of $R\left(\frac{d}{dt}\right)w = 0$ as the behavior, we take the $\mathscr{C}^\infty(\mathbb{R},\mathbb{R})$- solutions *of compact support*. Then there are only two cases: either $\mathscr{B} = \{0\},$ or $\mathscr{B} = \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$. Therefore, if we had taken the $\mathscr{C}^\infty(\mathbb{R},\mathbb{R})$- solutions of compact support as the behavior, the $1 \leftrightarrow 1$ relation with the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]$ fails.

This shows that the structure theorem is crucially dependent on the solution concept used. The theory of LTIDSs does not only depend on *algebra*, through submodules and the like, but also on *analysis*, through the sulotion concept of differential equations used.

Inclusion of behaviors

Let $\mathscr{B}_1 = \texttt{kernel}\left(R_1\left(\frac{d}{dt}\right)\right), \mathscr{B}_2 = \texttt{kernel}\left(R_2\left(\frac{d}{dt}\right)\right)$**. Then**

$$[\![\mathscr{B}_1 \subseteq \mathscr{B}_2]\!] \Leftrightarrow [\![\exists\, F \in \mathbb{R}[\xi]^{\bullet\times\bullet} \text{ such that } R_2 = FR_1]\!]$$

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Therefore,

$$[\![\mathscr{B}_1 = \mathscr{B}_2]\!] \Leftrightarrow [\![\exists\, F_1, F_2 \in \mathbb{R}[\xi]^{\bullet\times\bullet} \text{ such that } R_1 = F_2 R_2, R_2 = F_1 R_1]\!].$$

In particular,

$$[\![\mathscr{B}_1 = \mathscr{B}_2]\!] \text{ if } [\![\exists\, U \in \mathbb{R}[\xi]^{\bullet\times\bullet} \text{ unimodular such that } R_1 = U R_2]\!].$$

Minimal kernel representations

The representation $R\left(\frac{d}{dt}\right)w = 0$ **of** $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **is said to be a minimal kernel representation if, among all kernel representations of** $\mathscr{B}$**,** R **has a minimal number of rows.**

Minimal kernel representations

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Theorem

Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$. The following are equivalent.

1. $R\left(\frac{d}{dt}\right)w = 0$ is a minimal kernel representation of $\mathscr{B}$.
2. The rows of R are linearly independent.
3. R has full row rank.

All minimal kernel representations of $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ are generated from one, $R\left(\frac{d}{dt}\right)w = 0$, by the transformation group

$$R \overset{U \text{ unimodular}}{\longmapsto} UR$$

Follows immediately from the structure theorem.

Free variables

Free variables

Let $\mathbb{I} = \{i_1, i_2, \ldots, i_{|\mathbb{I}|}\} \subseteq \{1, 2, \ldots, \mathtt{w}\}$**. Define, for** $w = (w_1, w_2, \ldots, w_{\mathtt{w}}) \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **and** $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$**,**

$$\Pi_{\mathbb{I}} w := (w_{i_1}, w_{i_2}, \ldots, w_{i_{|\mathbb{I}|}}),$$

$$\Pi_{\mathbb{I}} \mathscr{B} := \{\Pi_{\mathbb{I}} w \mid w \in \mathscr{B}\}.$$

By the elimination theorem (see Lecture III),

$$[\![\mathscr{B} \in \mathscr{L}^{\mathtt{w}}]\!] \Rightarrow [\![\Pi_{\mathbb{I}} \mathscr{B} \in \mathscr{L}^{|\mathbb{I}|}]\!].$$

Free variables

The variables $\{w_{i_1}, w_{i_2}, \ldots, w_{i_{|\mathbb{I}|}}\}$ **are said to be free in** $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **if**

$$\Pi_{\mathbb{I}}\mathscr{B} = \mathscr{C}^{\infty}\left(\mathbb{R}, \mathbb{R}^{|\mathbb{I}|}\right),$$

i.e., if $\mathscr{B}$ **does not constrain the variables** $\{w_{i_1}, w_{i_2}, \ldots, w_{i_{|\mathbb{I}|}}\}$**.**

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The variables $\{w_{i_1}, w_{i_2}, \dots, w_{i_{|\mathbb{I}|}}\}$ **are said to be maximally free in** $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **if**

- $\Pi_{\mathbb{I}}\mathscr{B} = \mathscr{C}^{\infty}\left(\mathbb{R}, \mathbb{R}^{|\mathbb{I}|}\right)$,
- $[\![\mathbb{I}' = \{i'_1, i'_2, \dots, i'_{|\mathbb{I}'|}\} \subseteq \{1, 2, \dots, \mathtt{w}\}, \mathbb{I} \subseteq \mathbb{I}', \mathbb{I} \neq \mathbb{I}']\!]$
  $\Rightarrow [\![$**the variables** $\{w_{i'_1}, w_{i'_2}, \dots, w_{i'_{|\mathbb{I}'|}}\}$ **are not free in** $\mathscr{B}]\!]$ **.**

In words, these variables are unconstrained, but adding any other variable results in a constrained set of variables.

Free variables in LTIDSs

Partition $w = (w_1, w_2), w_1 : \mathbb{R} \to \mathbb{R}^{\mathtt{w}_1}, w_2 : \mathbb{R} \to \mathbb{R}^{\mathtt{w}_2}$**. Let** $R_1\left(\frac{d}{dt}\right) w_1 = R_2\left(\frac{d}{dt}\right) w_2$, **be a minimal kernel representation of** $\mathscr{B} \in \mathscr{L}^{\mathtt{w}_1+\mathtt{w}_2}$**.**

<u>Proposition 5</u>:

1. $[\![w_2$ **is free in** $\mathscr{B}]\!] \Leftrightarrow [\![R_1$ **has full row rank** $]\!]$.
2. $[\![w_2$ **is maximally free in** $\mathscr{B}]\!]$
   $\Leftrightarrow [\![R_1$ **is square and** $\mathtt{determinant}(R_1) \neq 0]\!]$.

Note that, by Proposition 4 from the section on differential operators, 2. is equivalent to:

2'. w_2 **is free and the elements of the form** $(w_1, 0) \in \mathscr{B}$ **form a finite-dimensional subspace.**

Proof in telegram-style

1. ($\Leftarrow$) $R_1\left(\frac{d}{dt}\right)$ **is surjective, hence** w_2 **is free.**

1. ($\Rightarrow$) If R_1 **does not have full row rank, then after pre-multiplication by a unimodular matrix, the minimal kernel representation looks like**

$$\begin{bmatrix} R_1'\left(\frac{d}{dt}\right) \\ 0_{\mathtt{rank}(R_1)\times\mathtt{w}_1} \end{bmatrix} w_1 = \begin{bmatrix} R_2'\left(\frac{d}{dt}\right) \\ R_2''\left(\frac{d}{dt}\right) \end{bmatrix} w_2,$$

with R_2'' **of full row rank. Therefore** w_2 **satisfies** $R_w''\left(\frac{d}{dt}\right) w_2 = 0$ **and is hence not free.**

Proof in telegram-style

2. $(\Leftarrow)$ w_2 **is free, by 1. Moreover, the elements of the form** $(w_1, 0) \in \mathscr{B}$ **form a finite-dimensional subspace, and therefore there are no additional free variables.**

2. $(\Rightarrow)$ **By 1.** R_1 **has full row rank. If** R_1 **is 'wide' (less rows than columns), then it possible to delete a column from** R_1 **and add it to** R_2**, and preserve the full row rank property. Then by 1.** w_2 **augmented with the variable from** w_1 **corresponding to the deleted column remains free.**

Examples

- **Consider**

$$r_1\left(\frac{d}{dt}\right)w_1 = r_2\left(\frac{d}{dt}\right)w_2,$$

with $r_1, r_2 \in \mathbb{R}[\xi]$**,** $r_1, r_2 \neq 0$**, and** $w_1, w_2 : \mathbb{R} \to \mathbb{R}$**.**
Then both w_1 **and** w_2 **are maximally free.**

Examples

- **Consider**
$$r_1\left(\frac{d}{dt}\right)w_1 = r_2\left(\frac{d}{dt}\right)w_2,$$
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Then both w_1 **and** w_2 **are maximally free.**

- **Consider**
$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}.$$

u **is free, and since the set of** y**'s corresponding to** $u = 0$ **is finite-dimensional, it is maximally free. Therefore the** (u, y) **behavior has a kernel representation**

$$P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u,$$

with P **square and** $\texttt{determinant}(P) \neq 0$.

Inputs and outputs

Input/output partition

Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ and $w = (u, y)$ with u maximally free in $\mathscr{B}$.
Then u is said to be input and y is said to be output in $\mathscr{B}$.
The corresponding partition $w = (u, y)$ is said to be an input/output partition for $\mathscr{B}$.

It follows from Proposition 5 that $w = (u, y)$ is an input/output partition if and only if $\mathscr{B}$ has a minimal kernel representation

$$P\left(\frac{d}{dt}\right) y = Q\left(\frac{d}{dt}\right) u,$$

with P square and $\mathtt{determinant}(P) \neq 0$.

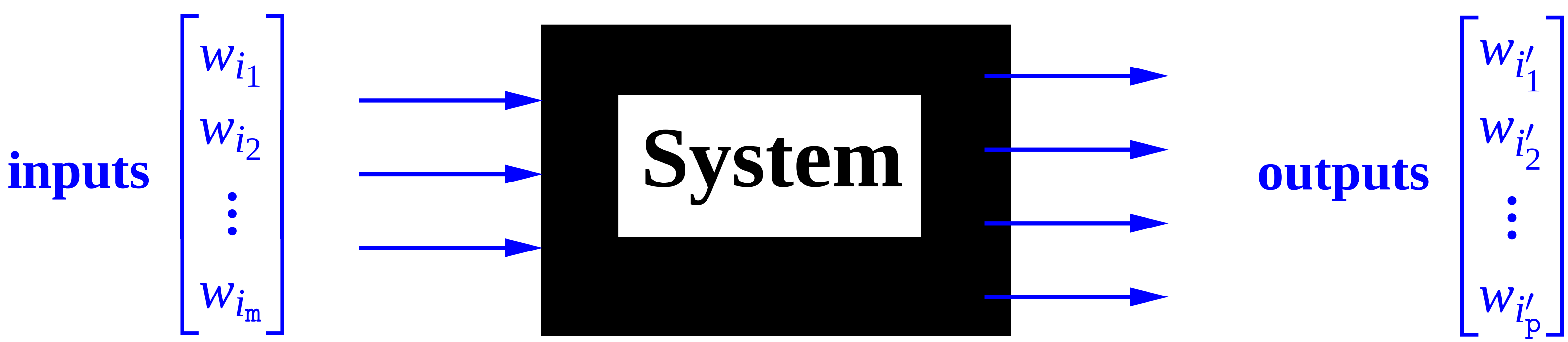
Input/output partition

Theorem

Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$. Then there exists a partition of the index set $\{1, 2, \ldots, \mathtt{w}\}$ into two parts, $\{i_1, i_2, \ldots, i_{\mathtt{m}}\}$ and $\{i'_1, i'_2, \ldots, i'_{\mathtt{p}}\}$ such that

$$u = (w_{i_1}, w_{i_2}, \ldots, w_{i_{\mathtt{m}}}), \quad y = (w_{i'_1}, w_{i'_2}, \ldots, w_{i'_{\mathtt{p}}})$$

is an input/output partition for $\mathscr{B}$.



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Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$. Then there exists a partition of the index set $\{1, 2, \ldots, \mathtt{w}\}$ into two parts, $\{i_1, i_2, \ldots, i_{\mathtt{m}}\}$ and $\{i'_1, i'_2, \ldots, i'_{\mathtt{p}}\}$ such that

$$u = (w_{i_1}, w_{i_2}, \ldots, w_{i_{\mathtt{m}}}), \quad y = (w_{i'_1}, w_{i'_2}, \ldots, w_{i'_{\mathtt{p}}})$$

is an input/output partition for $\mathscr{B}$.

Proof: Let $R\left(\frac{d}{dt}\right) w = 0$ be a minimal kernel representation of $\mathscr{B}$. Choose $\{i'_1, i'_2, \ldots, i'_{\mathtt{p}}\}$ such that the columns $\{i'_1, i'_2, \ldots, i'_{\mathtt{p}}\}$ of R form a square and nonsingular matrix.

Input/output partition

Theorem

Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$. Then there exists a partition of the index set $\{1, 2, \ldots, \mathtt{w}\}$ into two parts, $\{i_1, i_2, \ldots, i_{\mathtt{m}}\}$ and $\{i'_1, i'_2, \ldots, i'_{\mathtt{p}}\}$ such that

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is an input/output partition for $\mathscr{B}$.

It follows from the construction used in this proof that an input/output partition for $\mathscr{B}$ is in general not unique. However, the *number* of input and output components is uniquely determined by $\mathscr{B}$.

Integer invariants

$\mathtt{m} : \mathscr{L}^\bullet \to \mathbb{N}$, $\mathtt{m}(\mathscr{B}) :=$ **the number of input components in** $\mathscr{B}$,

$\mathtt{p} : \mathscr{L}^\bullet \to \mathbb{N}$, $\mathtt{p}(\mathscr{B}) :=$ **the number of output components in** $\mathscr{B}$,

$\mathtt{w} : \mathscr{L}^\bullet \to \mathbb{N}$, $\mathtt{w}(\mathscr{B}) :=$ **the number of real variables in** $\mathscr{B}$.

Of course $\mathtt{m} + \mathtt{p} = \mathtt{w}$**.**

Note the following formulas for $\mathtt{p}$**:**

$$\mathtt{p}(\mathscr{B}) = \mathtt{\textbf{dimension}}(\mathscr{N}_{\mathscr{B}}).$$

$$\mathtt{p}(\mathscr{B}) = \mathtt{\textbf{rowdimension}}(R)$$

with $R\left(\frac{d}{dt}\right)w = 0$ **a minimal kernel representation of** $\mathscr{B}$**.**

The transfer function

Let $w = (u, y)$ **be an input/output partition of** $\mathscr{B} \in \mathscr{L}^{\mathtt{m}(\mathscr{B}) \times \mathtt{p}(\mathscr{B})}$**, with kernel representation**

$$P\left(\frac{d}{dt}\right) y = Q\left(\frac{d}{dt}\right) u.$$

The $\mathtt{m}(\mathscr{B}) \times \mathtt{p}(\mathscr{B})$ **matrix of real rational functions**

$$G = P^{-1} Q$$

is called the transfer function corresponding to this input/output partition.

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Note that for each $\lambda \in \mathbb{C}$**, not a pole of** G**, and for each** $\mathrm{u}_\lambda \in \mathbb{C}^{\mathtt{m}(\mathscr{B})}$**, the exponential trajectory**

$$t \mapsto \left(\mathrm{u}_\lambda e^{\lambda t}, \mathrm{y}_\lambda e^{\lambda t}\right), \ \textbf{with } \mathrm{y}_\lambda = G(\lambda)\mathrm{u}_\lambda,$$

belongs to $\mathscr{B}$ **(complexified).**

The transfer function can be defined by means of this formula for the exponential response.

Proper transfer functions

The real rational function $f = \frac{n}{d} \in \mathbb{R}(\xi), n, d \in \mathbb{R}[\xi]$ **is said to be** $[\![$ **proper** $]\!]$ $:\Leftrightarrow$ $[\![\texttt{degree}(d) \geq \texttt{degree}(n)]\!]$.

A matrix of real rational functions is said to be $[\![$ **proper** $]\!]$ $:\Leftrightarrow$ $[\![$**each element of the matrix is proper**$]\!]$.

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Theorem

Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$**. Then there exists a partition of the index set** $\{1, 2, \ldots, \mathtt{w}\}$ **into two parts,** $\{i_1, i_2, \ldots, i_{\mathtt{m}(\mathscr{B})}\}$ **and** $\{i'_1, i'_2, \ldots, i'_{\mathtt{p}(\mathscr{B})}\}$ **such that**

$$u = (w_{i_1}, w_{i_2}, \ldots, w_{i_{\mathtt{m}(\mathscr{B})}}), \quad y = (w_{i'_1}, w_{i'_2}, \ldots, w_{i'_{\mathtt{p}(\mathscr{B})}})$$

is an input/output partition for $\mathscr{B}$ **with a proper transfer function.**

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is an input/output partition for $\mathscr{B}$ with a proper transfer function.

Proof: When selecting $\mathtt{p}(\mathscr{B})$ columns of R corresponding to a minimal kernel representation $R\left(\frac{d}{dt}\right) w = 0$ of $\mathscr{B}$, choose the columns $\{i'_1, i'_2, \ldots, i'_{\mathtt{p}(\mathscr{B})}\}$ such that the determinant of the matrix formed by these columns has largest degree among all $\mathtt{p}(\mathscr{B}) \times \mathtt{p}(\mathscr{B})$ submatrices of R.

Significance of a proper transfer functions

- **For continuous-time system the significance of a proper transfer function lies in the fact that**
  the output is at least as smooth as the input.
  Unfortunately, this cannot be illustrated in our $\mathscr{C}^\infty$-setting. However, if the behavior is defined as a set of distributions, properness comes down to the implication

$$[\![(u,y) \in \mathscr{B}, u \in \mathscr{C}^{\mathtt{k}}\left(\mathbb{R}, \mathbb{R}^{\mathtt{m}(\mathscr{B})}\right)]\!] \Rightarrow [\![y \in \mathscr{C}^{\mathtt{k}}\left(\mathbb{R}, \mathbb{R}^{\mathtt{p}(\mathscr{B})}\right)]\!].$$

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- **For discrete-time systems, properness implies that**
  the output does not anticipate the input.
  This is made precise in Exercise II.5.

Examples

- **Consider**

$$\frac{d}{dt}x = Ax + Bu, \;\; y = Cx + Du, \;\; w = \begin{bmatrix} u \\ y \end{bmatrix}.$$

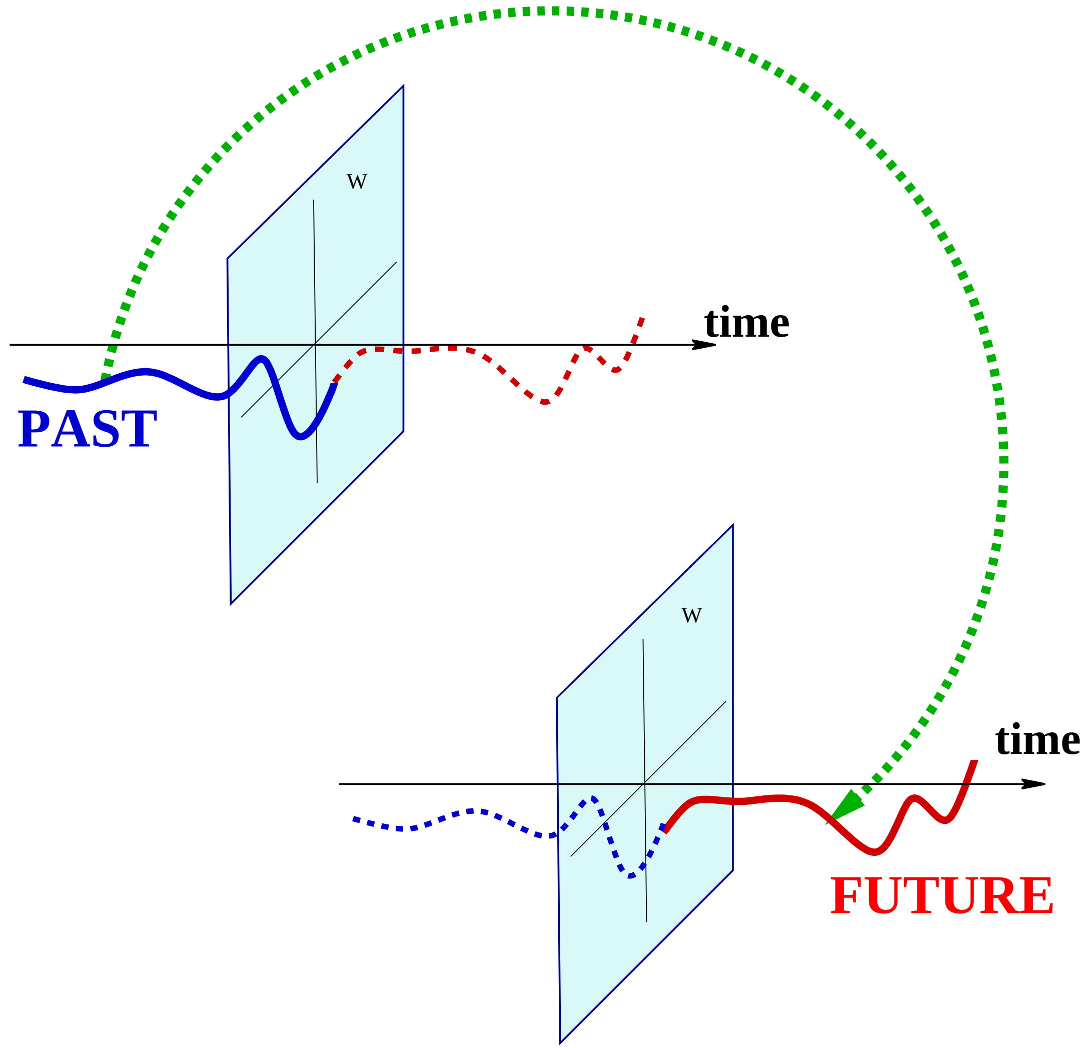
In order to compute the transfer function, it is perhaps easiest to proceed via the exponential response. This yields

$$G(\xi) = D + C(I\xi - A)^{-1}B$$

for the transfer function. This matrix of rational functions is proper, hence y is at least as smooth as u.

Autonomous LTIDSs

Reminder



autonomous :⇔ the past implies the future.

Characterization of autonomous LTIDSs

Theorem

The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$.

- $\mathscr{B}$ **is autonomous.**
- $\mathscr{B}$ **is a finite-dimensional subspace of** $\mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.
- $\mathscr{B}$ **has a kernel representation** $R\left(\frac{d}{dt}\right)w = 0$ **with** R **of rank** $\mathtt{w}$.
- $\mathscr{B}$ **has a minimal kernel representation** $R\left(\frac{d}{dt}\right)w = 0$ **with** R **square and** $\mathtt{determinant}(R) \neq 0$.
- $\mathtt{m}(\mathscr{B}) = 0$**, equivalently,** $\mathtt{p}(\mathscr{B}) = \mathtt{w}(\mathscr{B})$.

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- $\mathtt{m}(\mathscr{B}) = 0$**, equivalently,** $\mathtt{p}(\mathscr{B}) = \mathtt{w}(\mathscr{B})$**.**

The proof follows readily from the Smith form and Propositions 1 and 3 of the section on differential operators.

Characterization of autonomous LTIDSs

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- $\mathscr{B}$ **has a minimal kernel representation** $R\left(\frac{d}{dt}\right) w = 0$ **with** R **square and** $\mathtt{determinant}(R) \neq 0$**.**
- $\mathtt{m}(\mathscr{B}) = 0$**, equivalently,** $\mathtt{p}(\mathscr{B}) = \mathtt{w}(\mathscr{B})$**.**

With R **minmal, there holds, for** $\mathtt{w} = 1$**,**
$\mathtt{dimension}(\mathscr{B}) = \mathtt{degree}(R)$, **while for** $\mathtt{w} > 1$**,**
$\mathtt{dimension}(\mathscr{B}) = \mathtt{degree}(\mathtt{determinant}(R))$.

Autonomous LTIDSs

Each trajectory w of an autonomous $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ is a sum of products of a polynomial and an exponential in the complex case,

$$w(t) = \pi_1(t)e^{\lambda_1 t} + \pi_2(t)e^{\lambda_2 t} + \cdots + \pi_{\mathtt{r}}(t)e^{\lambda_{\mathtt{r}} t},$$

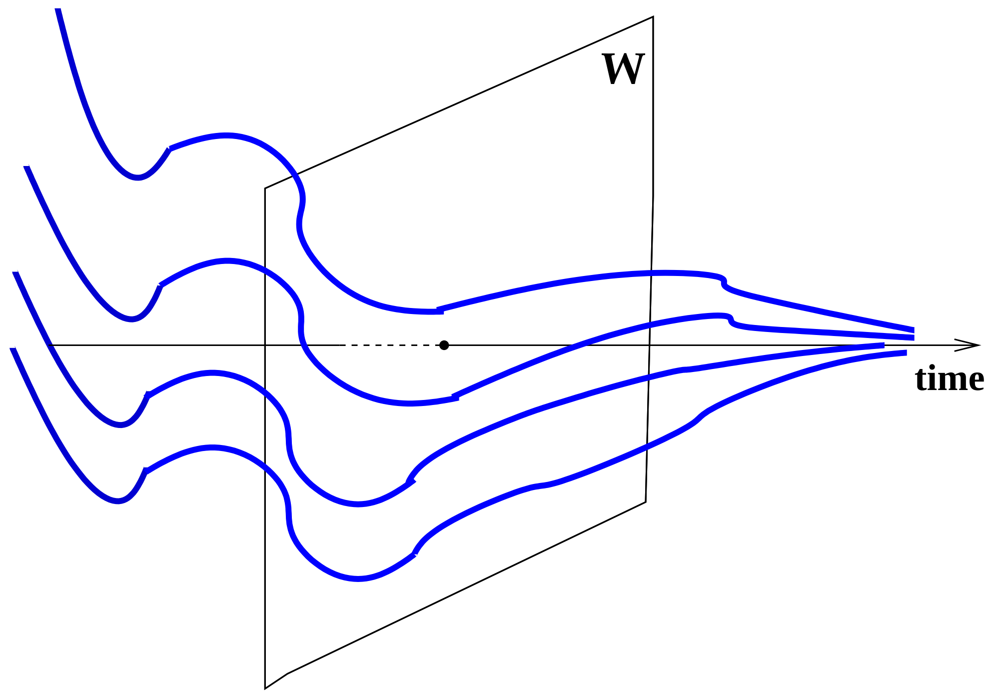
with $\pi_{\mathtt{k}} \in \mathbb{C}[\xi]^{\mathtt{w}}$ and $\lambda_{\mathtt{k}} \in \mathbb{C}$. In the real case, it is a sum of products of a polynomial, an exponential, and a trigonometric function,

$$\begin{aligned} w(t) = \pi_1'(t)e^{\lambda_1 t}\mathbf{cos}(\omega_1 t) + \pi_1''(t)e^{\lambda_1 t}\mathbf{sin}(\omega_1 t) \\ + \pi_2'(t)e^{\lambda_2 t}\mathbf{cos}(\omega_2 t) + \pi_2''(t)e^{\lambda_2 t}\mathbf{sin}(\omega_2 t) \\ + \cdots + \pi_{\mathtt{r}}'(t)e^{\lambda_{\mathtt{r}} t}\mathbf{cos}(\omega_{\mathtt{r}} t) + \pi_{\mathtt{r}}''(t)e^{\lambda_{\mathtt{r}} t}\mathbf{sin}(\omega_{\mathtt{r}} t), \end{aligned}$$

with $\pi_{\mathtt{k}}', \pi_{\mathtt{k}}'' \in \mathbb{R}[\xi]^{\mathtt{w}}, \lambda_{\mathtt{k}} \in \mathbb{R}$, and $\omega_{\mathtt{k}} \in \mathbb{R}$.

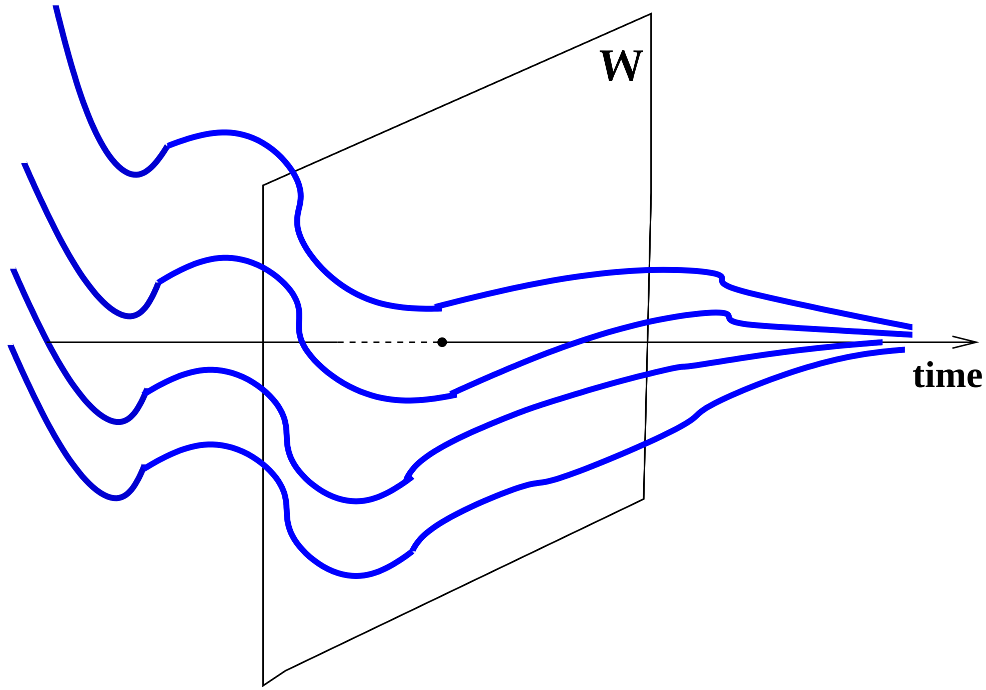
Stability

Reminder



stability :⇔ all trajectories go to 0.

Reminder



stability :$\Leftrightarrow$ all trajectories go to 0.

For $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$, there holds $[\![\mathscr{B}$ stable$]\!] \Rightarrow [\![\mathscr{B}$ autonomous$]\!]$.

Stability of LTIDSs

Theorem

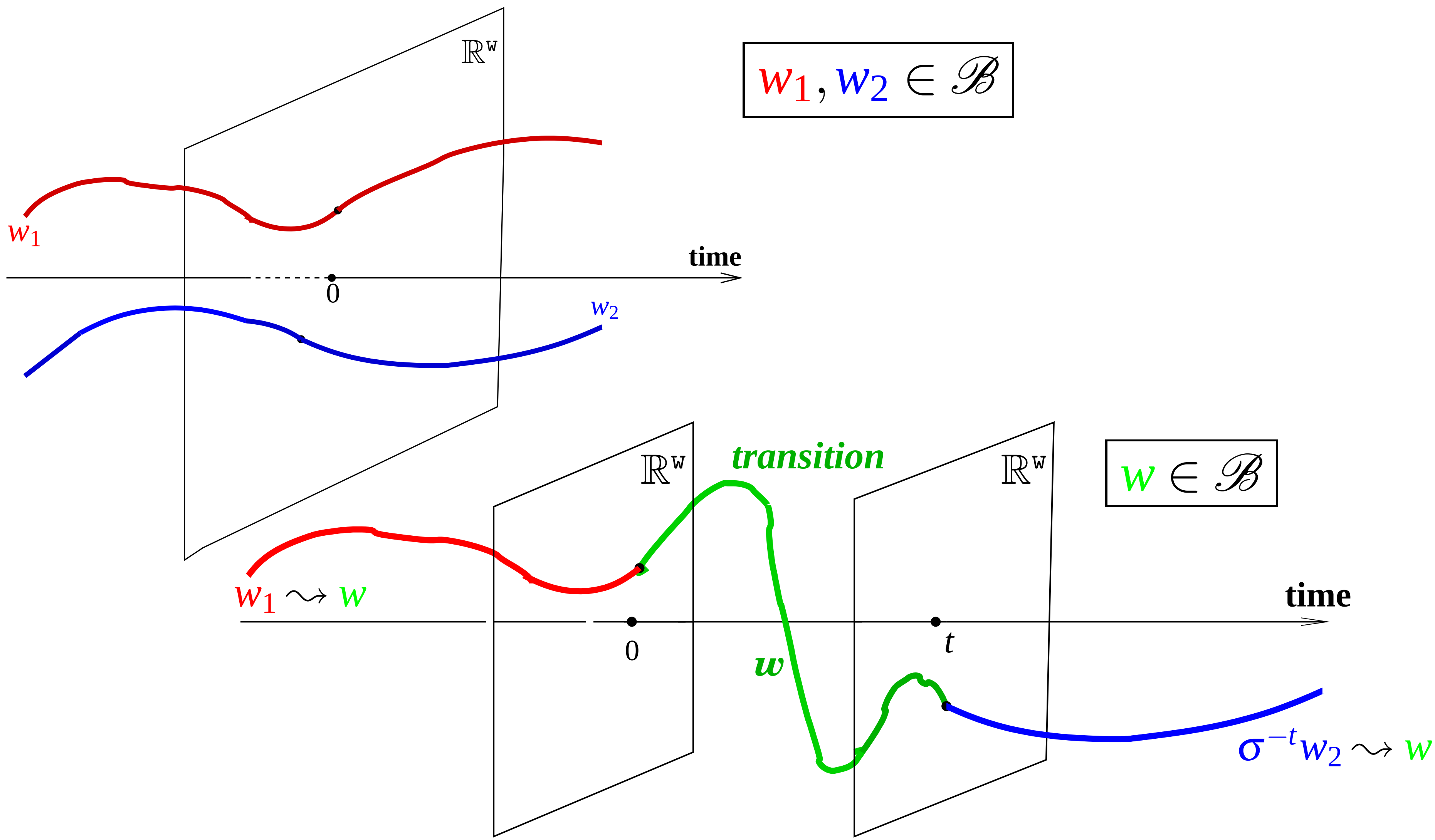
The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$.

- $\mathscr{B}$ **is stable.**
- **Every exponential trajectory** $t \mapsto e^{\lambda t} a, \;\; a \in \mathbb{C}^{\mathtt{w}}$, **in** $\mathscr{B}$ **(complexified) has** $\mathtt{Real}(\lambda) < 0$.
- $\mathscr{B}$ **has a kernel representation** $R\left(\frac{d}{dt}\right) w = 0$ **with** $\mathtt{rank}(R) = \mathtt{w}$ **and** $[\![\mathtt{rank}(R(\lambda)) < \mathtt{w}, \lambda \in \mathbb{C}]\!] \Rightarrow [\![\mathtt{Real}(\lambda) < 0]\!]$.
- $\mathscr{B}$ **has a minimal kernel representation** $R\left(\frac{d}{dt}\right) w = 0$ **with** R **Hurwitz.**

A polynomial $\in \mathbb{C}[\xi]$ is said to be Hurwitz if all its roots are in $\{\lambda \in \mathbb{C} \mid \mathtt{Real}(\lambda) < 0\}$. $P \in \mathbb{C}[\xi]^{\mathtt{n} \times \mathtt{n}}$ is said to be Hurwitz if it is square and $\mathtt{determinant}(R)$ is Hurwitz.

Controllability and stabilizability

Reminder



controllability :⇔ concatenability of trajectories after a delay

Controllability of LTIDSs

Theorem

The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$.

1. $\mathscr{B}$ **is controllable.**
2. $\mathscr{B}$ **has a kernel representation** $R\left(\frac{d}{dt}\right)w = 0$ **and** $R(\lambda)$ **has the same rank for all** $\lambda \in \mathbb{C}$.
3. $\mathscr{N}_{\mathscr{B}}$, **the** $\mathbb{R}[\xi]$**-module of annihilators of** $\mathscr{B}$**, is closed.**
4. $\mathscr{B}$ **has a direct summand, i.e., there exists** $\mathscr{B}' \in \mathscr{L}^{\mathtt{w}}$ **such that** $\mathscr{B} \oplus \mathscr{B}' = \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

The closure of the $\mathbb{R}[\xi]$**-submodule** $\mathscr{M}$ **of** $\mathbb{R}[\xi]^{\mathtt{w}}$ **is defined as**

$$\mathscr{M}^{\text{closure}} := \{\bar{m} \in \mathbb{R}[\xi]^{\mathtt{w}} \mid \exists\, \pi \in \mathbb{R}[\xi], \pi \neq 0, \text{ and } m \in \mathscr{M} \text{ such that } m = \pi\bar{m}\}.$$

$\mathscr{M}$ **is said to be closed if** $\mathscr{M} = \mathscr{M}^{\text{closure}}$.

Proof in telegram-style

First prove that if $U \in \mathbb{R}[\xi]^{\mathtt{w}\times\mathtt{w}}$ **is unimodular, then** $\mathscr{B}$ **is controllable if and only if** $U\left(\frac{d}{dt}\right)\mathscr{B}$ **is controllable.**
Consequently, we may assume that $\mathscr{B}$ **has a minimal kernel representation with** R **in Smith form,**

$$R = \begin{bmatrix} \mathbf{diag}(d_1, d_2, \ldots, d_{\mathtt{r}}) & 0_{\mathtt{r}\times(\mathtt{w}-\mathtt{r})} \end{bmatrix}.$$

1. $\Leftrightarrow$ **2.**

Observe, using the theory of autonomous systems, that $\mathscr{B}$ **is controllable if and only if all the invariant polynomials** $d_1, d_2, \ldots, d_{\mathtt{r}}$ **of** R **are equal to one. Equivalently, if and only if 2. holds.**

Proof in telegram-style

Consequently, we may assume that $\mathscr{B}$ has a minimal kernel representation with R in Smith form,

$$R = \begin{bmatrix} \mathbf{diag}(d_1, d_2, \ldots, d_{\mathtt{r}}) & 0_{\mathtt{r}\times(\mathtt{w}-\mathtt{r})} \end{bmatrix}.$$

2. $\Leftrightarrow$ 3.

Observe that $\mathscr{N}_{\mathscr{B}} = \begin{bmatrix} \mathbb{R}[\xi]\, d_1 & \cdots & \mathbb{R}[\xi]\, d_{\mathtt{r}} & 0 & \cdots & 0 \end{bmatrix}$**.**
Hence $\mathscr{N}_{\mathscr{B}}$ is closed if and only if all the invariant polynomials $d_1, d_2, \ldots, d_{\mathtt{r}}$ of R are equal to one. Equivalently, if and only if 2. holds.

Proof in telegram-style

Consequently, we may assume that $\mathscr{B}$ has a minimal kernel representation with R in Smith form,

$$R = \begin{bmatrix} \mathbf{diag}(d_1, d_2, \ldots, d_{\mathtt{r}}) & 0_{\mathtt{r}\times(\mathtt{w}-\mathtt{r})} \end{bmatrix}.$$

3. $\Rightarrow$ 4.

Take for $\mathscr{B}'$ the system with kernel representation

$R'\left(\frac{d}{dt}\right) w = 0$**, with** $R' = \begin{bmatrix} 0_{\mathtt{w}-\mathtt{r}\times\mathtt{r}} & I_{(\mathtt{w}-\mathtt{r})\times(\mathtt{w}-\mathtt{r})} \end{bmatrix}.$

Proof in telegram-style

4. $\Rightarrow$ 3.

Note that $[\![\mathscr{B} \oplus \mathscr{B}' = \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})]\!] \Leftrightarrow [\![\mathscr{N}_{\mathscr{B}} \oplus \mathscr{N}_{\mathscr{B}'} = \mathbb{R}[\xi]^{1 \times \mathtt{w}}]\!]$**.**
Let $R'\left(\frac{d}{dt}\right) w = 0$ **be a minimal kernel representation of** $\mathscr{B}'$**.**

Then $\mathscr{N}_{\mathscr{B}} \oplus \mathscr{N}_{\mathscr{B}'} = \mathbb{R}[\xi]^{\mathtt{w}}$ **implies that the rows of** $\begin{bmatrix} R \\ R' \end{bmatrix}$ **form a basis for** $\mathbb{R}[\xi]^{1 \times \mathtt{w}}$**. Equivalently, that** $\begin{bmatrix} R \\ R' \end{bmatrix}$ **is unimodular.**
Hence that the invariant polynomials of R **are all equal to one.**

Examples

- **Consider the single-input/single output system**

$$p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix},$$

with $p, q \in \mathbb{R}[\xi]$. This system is controllable if and only if the polynomials p and q are coprime.

The problem of common factors in p and q and their interpretation has been a long-standing question in the field. Behavioral controllability demystifies this.
We now understand that common factors correspond exactly to lack of controllability.

Examples

- **Applying the controllability theorem and the relation between behavioral and state controllability discussed in Lecture I, shows that the system**

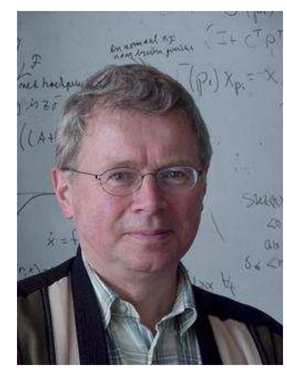
$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du, \quad w = \begin{bmatrix} u \\ y \\ x \end{bmatrix}$$

is controllable if and only if

$$\texttt{rank}\left(\begin{bmatrix} I_{\mathtt{n}\times\mathtt{n}}\lambda - A & \vdots & B \end{bmatrix}\right) = \mathtt{n} \quad \textbf{for all } \lambda \in \mathbb{C}.$$

This state version of this controllability test is called the PBH (Popov-Belevitch-Hautus) test. The controllability theorem is therefore a generalization of this classical result.

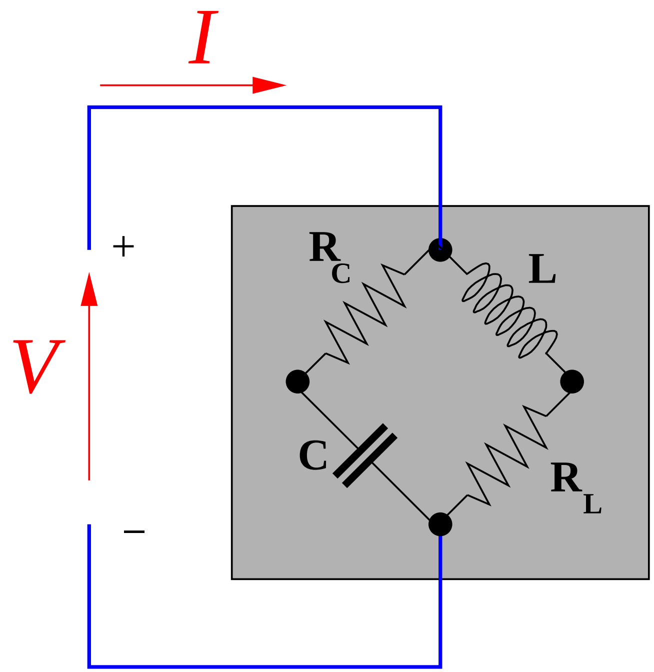
Vitold Belevitch (1921–1999)



Malo Hautus (1940–)

Examples

The port behavior of the RLC circuit (will be discussed in Lecture III)



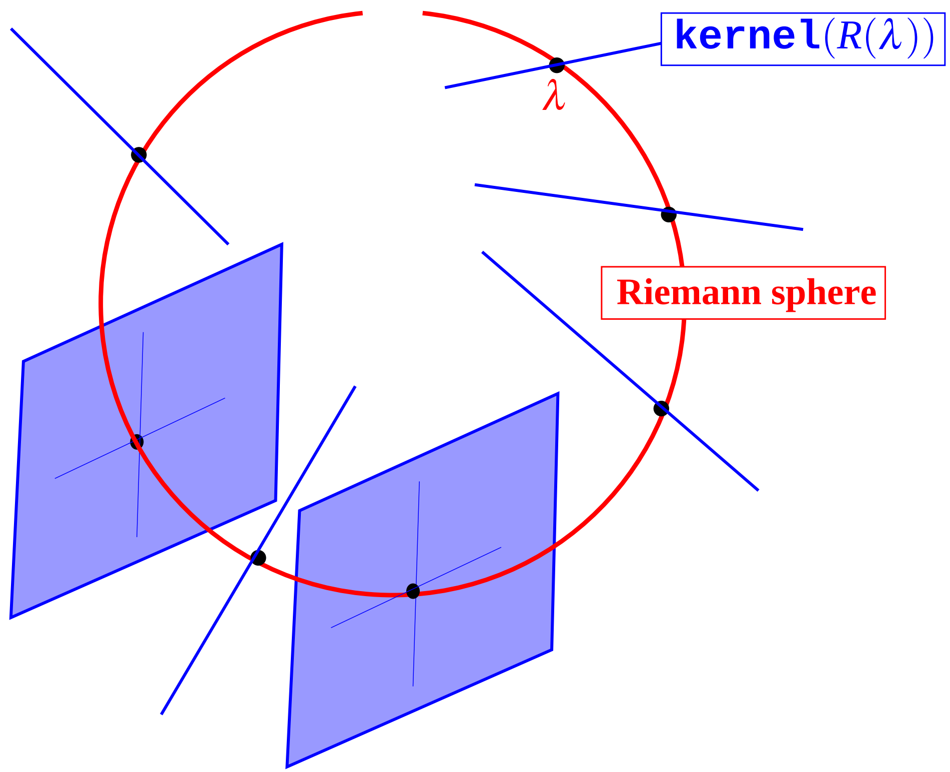
is controllable unless

$$CR_C = \frac{L}{R_L} \quad \textbf{and} \quad R_L = R_C.$$

This shows that lack of controllability can occur in non-degenerate physical systems.

Geometric interpretation of controllability

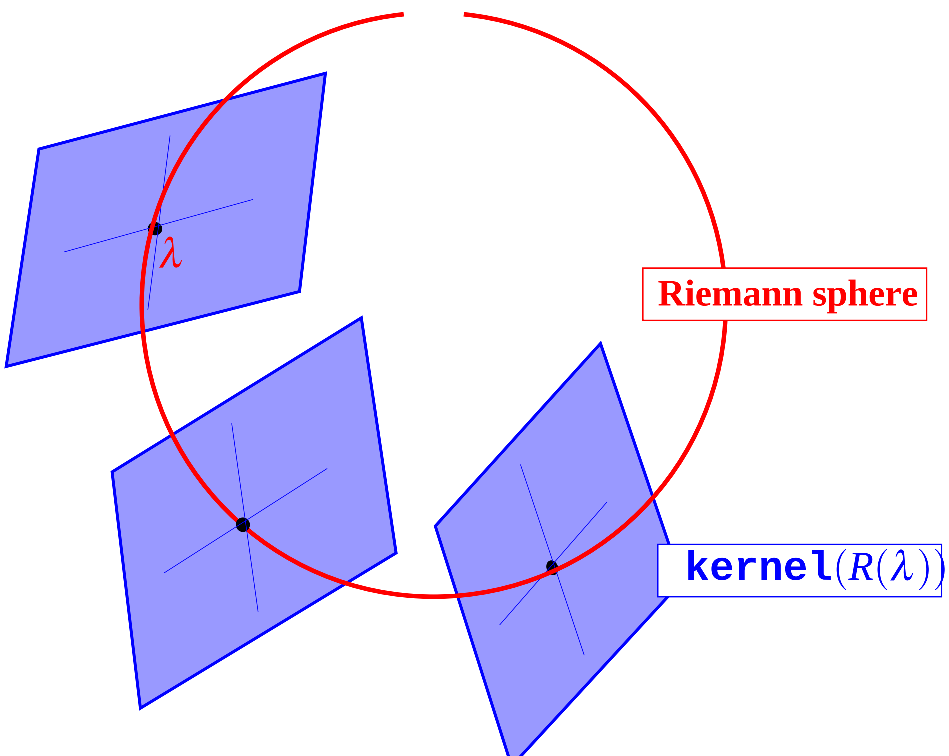
Attach to each point the Riemann sphere (think of the Riemann sphere as $\mathbb{C}$), $\texttt{kernel}(R(\lambda)), \lambda \in \mathbb{C}$. This associates with each $\lambda \in \mathbb{C}$, a linear subspace of $\mathbb{C}^{\mathtt{w}}$. In general, this yields a picture shown below. Since the dimension of the subspace attached may change, we obtain a 'sheaf'.



Bernhard Riemann
1826–1866

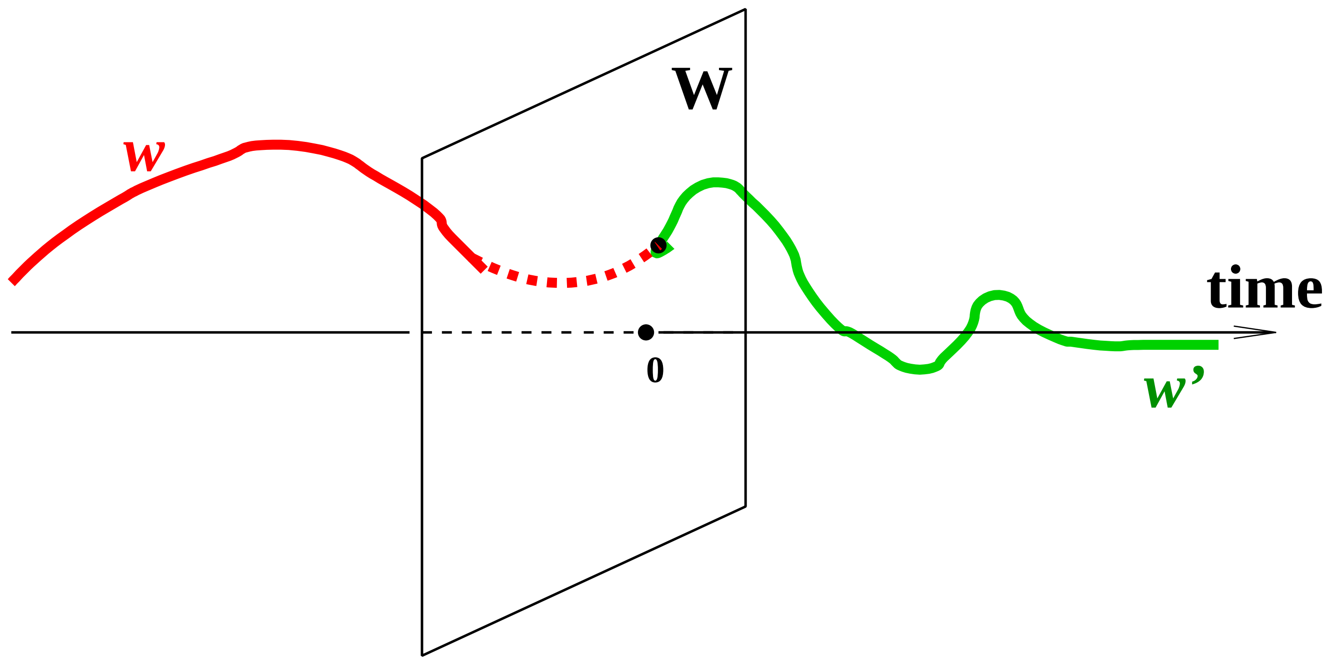
Geometric interpretation of controllability

Attach to each point the Riemann sphere (think of the Riemann sphere as $\mathbb{C}$), $\texttt{kernel}(R(\lambda)), \lambda \in \mathbb{C}$. This associates with each $\lambda \in \mathbb{C}$, a linear subspace of $\mathbb{C}^{\mathtt{w}}$.
The dimension of the subspace attached is constant, that is, we obtain a 'vector bundle' over the Riemann sphere, if and only if the system is controllable.



Bernhard Riemann
1826–1866

Reminder



stabilizability :$\Leftrightarrow$ all trajectories can be steered to 0.

Stabilizability of LTIDSs

Theorem

The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$.

1. **$\mathscr{B}$ is stabilizable.**
2. **$\mathscr{B}$ has a kernel representation $R\left(\frac{d}{dt}\right) w = 0$ and $R(\lambda)$ has the same rank for all $\lambda \in \{\lambda' \in \mathbb{C} \mid \mathtt{Real}(\lambda') \geq 0\}$.**

Proof in telegram-style

2. $\Rightarrow$ 1.

First assume that R is in Smith form. Prove that if 2. is satisfied, then the first $\mathtt{r}$ components of w are polynomial exponentials, with exponentials having negative real part, while the remaining components of w are free. Conclude stabilizability.

1. $\Rightarrow$ 2.

Conversely, if 2. is not satisfied, then there is a solution whose first component is an exponential with real part ≥ 0.
This exponential cannot be steered to zero, and the system is not stabilizable.

Proof in telegram-style

Next, consider general R's. The solutions are now of the form $w = U\left(\frac{d}{dt}\right) w'$ with $U \in \mathbb{R}[\xi]^{\mathtt{w}\times\mathtt{w}}$ unimodular, and w' a solution corresponding to the Smith form of R. The arguments extend, since polynomial exponentials are converted by $U\left(\frac{d}{dt}\right)$ to polynomial exponentials with the same exponential coefficients.

Controllability

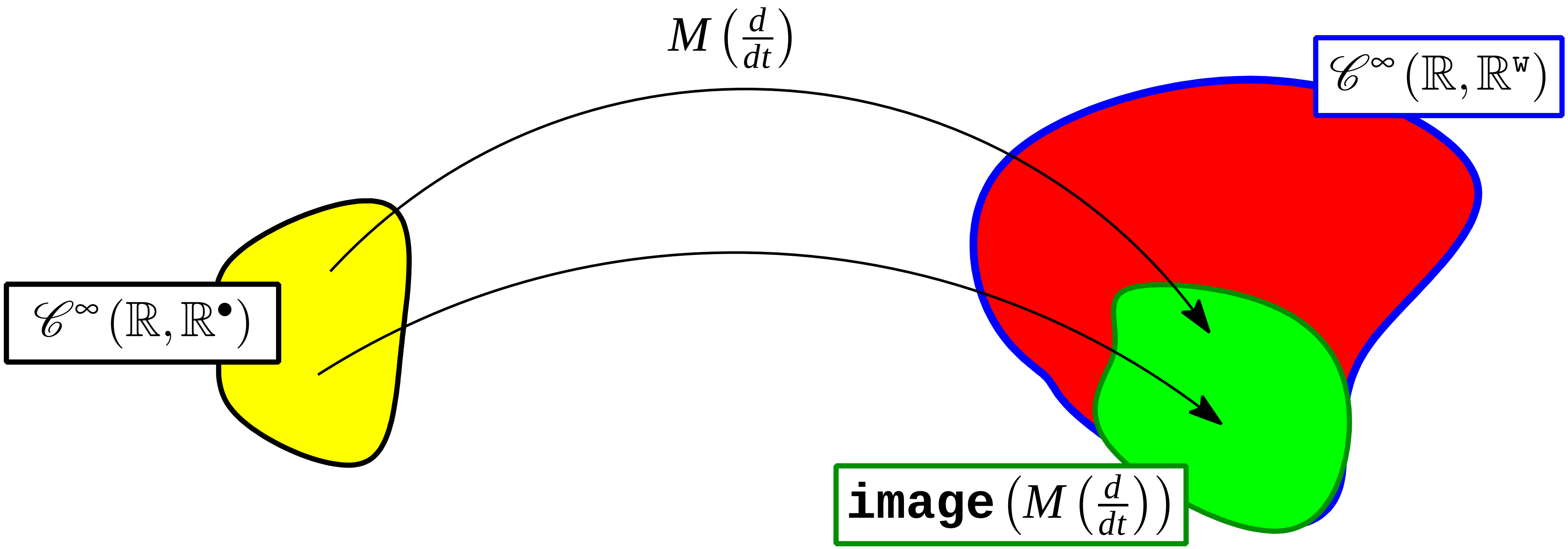
and

image representations

Images

We have seen plenty of kernels. It is time to discuss images.

$$w = M\left(\frac{d}{dt}\right)\ell.$$



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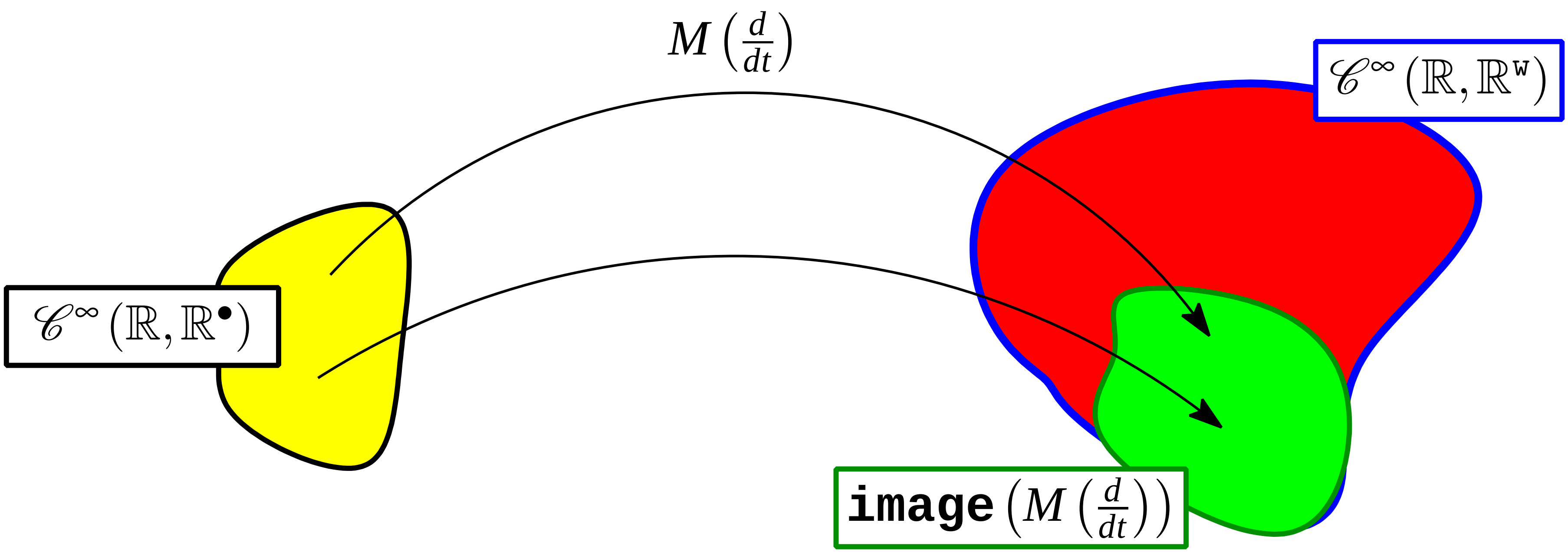


Elimination theorem (see Lecture III) $\Rightarrow$ an image is a kernel.

What is special about images ?

The image representation theorem

Theorem

The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$.

1. **$\mathscr{B}$ is controllable.**
2. **$\mathscr{B}$ has a image representation**

$$w = M\left(\frac{d}{dt}\right)\ell.$$

Images have a nice system theoretic interpretation!

Proof in telegram-style

1. $\Rightarrow$ 2.

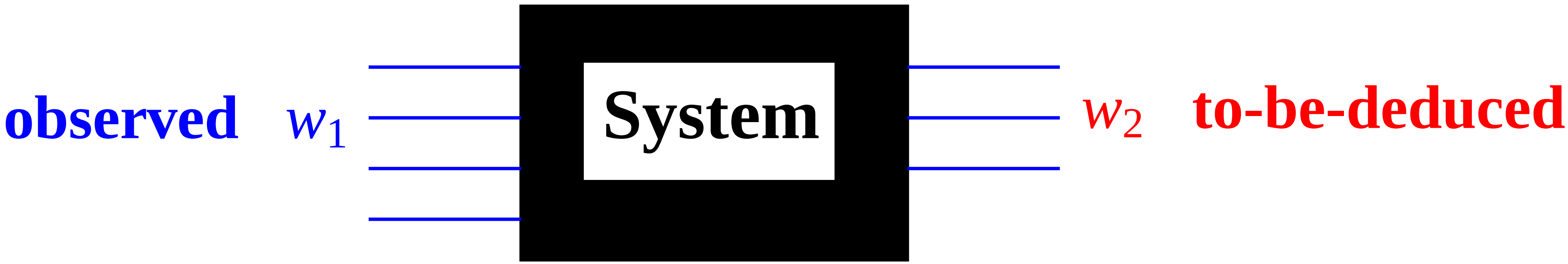
By controllability, the invariant polynomials of R, with $R\left(\frac{d}{dt}\right)w = 0$ a minimal kernel representation of $\mathscr{B}$, are equal to one. Therefore, $R = V\begin{bmatrix} I_{\mathtt{r}\times\mathtt{r}} & 0_{\mathtt{r}\times(\mathtt{w}-\mathtt{r})}\end{bmatrix}U$, with U, V unimodular. It follows that $w = U^{-1}\left(\frac{d}{dt}\right)\begin{bmatrix} 0_{\mathtt{r}\times(\mathtt{w}-\mathtt{r})} \\ I_{(\mathtt{w}-\mathtt{r})\times(\mathtt{w}-\mathtt{r})}\end{bmatrix}\ell$ is an image representation of $\mathscr{B}$.

2. $\Rightarrow$ 1.

The extended behavior $\{(w,\ell) \mid w = M\left(\frac{d}{dt}\right)\ell\}$ is controllable, since $\begin{bmatrix} I_{\mathtt{w}\times\mathtt{w}} & -M(\lambda)\end{bmatrix}$ has rank $\mathtt{w}$ for all $\lambda \in \mathbb{C}$.
This implies that the projection is controllable.

Observability and detectability

Reminder



observability :⇔ w_2 may be deduced from w_1.

!!! Knowing the laws of the system !!!

The observability theorem

Theorem

The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}_1+\mathtt{w}_2}$, $\mathscr{B} \subseteq \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1} \times \mathbb{R}^{\mathtt{w}_2})$.

1. w_2 **is observable from** w_1 **in** $\mathscr{B}$.
2. $\mathscr{B}$ **has a kernel representation** $R_1\left(\frac{d}{dt}\right) w_1 = R_2\left(\frac{d}{dt}\right) w_2$, **with** $\mathtt{rank}(R_2(\lambda)) = \mathtt{w}_2$ **for all** $\lambda \in \mathbb{C}$.
3. $\mathscr{B}$ **has a minimal kernel representation**

$$w_2 = F\left(\frac{d}{dt}\right) w_1, \quad R\left(\frac{d}{dt}\right) w_1 = 0.$$

Proof in telegram-style

1. $\Leftrightarrow$ 2.

$$[\![\textbf{Observability}]\!] \Leftrightarrow [\![R_2\left(\tfrac{d}{dt}\right) \textbf{ is injective}]\!] \Leftrightarrow [\![\textbf{2.}]\!]$$

Proof in telegram-style

1. $\Leftrightarrow$ 2.

$[\![$Observability$]\!]$ $\Leftrightarrow$ $[\![R_2\left(\frac{d}{dt}\right)$ is injective$]\!]$ $\Leftrightarrow$ $[\![$2.$]\!]$

2. $\Leftrightarrow$ 3.

$(\Leftarrow)$ is obvious. To prove $(\Rightarrow)$, observe that 2. implies that R_2 is of the form $R_2 = V \begin{bmatrix} I_{\mathtt{w}_2 \times \mathtt{w}_2} \\ 0_{\bullet \times \mathtt{w}_2} \end{bmatrix} U$, with V, U unimodular. Therefore $\mathscr{B}$ admits the kernel representation

$$R_1'\left(\frac{d}{dt}\right) w_1 = U\left(\frac{d}{dt}\right) w_2, \quad R_1''\left(\frac{d}{dt}\right) w_1 = 0,$$

leading to 3.

The detectability theorem

Theorem

The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}_1+\mathtt{w}_2}$, $\mathscr{B} \subseteq \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1} \times \mathbb{R}^{\mathtt{w}_2})$.

1. w_2 **is detectable from** w_1 **in** $\mathscr{B}$.
2. $\mathscr{B}$ **has a kernel representation** $R_1\left(\frac{d}{dt}\right) w_1 = R_2\left(\frac{d}{dt}\right) w_2$, **with** $\mathtt{rank}(R_2(\lambda)) = \mathtt{w}_2$ **for** $\lambda \in \{\lambda' \in \mathbb{C} \mid \mathtt{Real}(\lambda') \geq 0\}$.
3. $\mathscr{B}$ **has a minimal kernel representation**

$$H\left(\frac{d}{dt}\right) w_2 = F\left(\frac{d}{dt}\right) w_1, \quad R\left(\frac{d}{dt}\right) w_1 = 0,$$

with H **Hurwitz.**

The proof is analogous to that of the observability theorem.

The controllable part

The controllable part of a LTIDS

The controllable part of $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$, denoted by $\mathscr{B}_{\text{controllable}}$, is defined as

1. $\mathscr{B}_{\text{controllable}} \in \mathscr{L}^{\mathtt{w}}$,
2. $\mathscr{B}_{\text{controllable}} \subseteq \mathscr{B}$,
3. $[\![\mathscr{B}' \subseteq \mathscr{B}$ **and** $\mathscr{B}'$ **controllable**$]\!] \Rightarrow [\![\mathscr{B}' \subseteq \mathscr{B}_{\text{controllable}}]\!]$.

Hence $\mathscr{B}_{\text{controllable}}$ is the largest controllable system contained in $\mathscr{B}$.

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Hence $\mathscr{B}_{\text{controllable}}$ is the largest controllable system contained in $\mathscr{B}$.

Let $R\left(\frac{d}{dt}\right) w = 0$ be a minimal kernel representation of $\mathscr{B}$. **The polynomial matrix R can be factored as $R = FR'$, with $F \in \mathbb{R}[\xi]^{\mathtt{p}(\mathscr{B}) \times \mathtt{p}(\mathscr{B})}$ and with $R' \in \mathbb{R}[\xi]^{\mathtt{p}(\mathscr{B}) \times \mathtt{w}(\mathscr{B})}$ such that $R'(\lambda)$ has the same rank for all $\lambda \in \mathbb{C}$. Then $R'\left(\frac{d}{dt}\right) w = 0$ is a kernel representation of $\mathscr{B}_{\text{controllable}}$.**

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Hence $\mathscr{B}_{\text{controllable}}$ is the largest controllable system contained in $\mathscr{B}$.

There holds

$$\mathscr{B}_{\text{controllable}} = \overline{\mathscr{B}_{\text{compact}}}^{\mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}(\mathscr{B})})},$$

where $\mathscr{B}_{\text{compact}}$ denotes the set of compact support trajectories in $\mathscr{B}$, and $\overline{\mathscr{B}_{\text{compact}}}^{\mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}(\mathscr{B})})}$ the closure of $\mathscr{B}_{\text{compact}}$ in the $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}(\mathscr{B})})$-topology.

The controllable part of a LTIDS

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1. $\mathscr{B}_{\text{controllable}} \in \mathscr{L}^{\mathtt{w}}$,
2. $\mathscr{B}_{\text{controllable}} \subseteq \mathscr{B}$,
3. $[\![\mathscr{B}' \subseteq \mathscr{B}$ **and** $\mathscr{B}'$ **controllable**$]\!] \Rightarrow [\![\mathscr{B}' \subseteq \mathscr{B}_{\text{controllable}}]\!]$.

Hence $\mathscr{B}_{\text{controllable}}$ **is the largest controllable system contained in** $\mathscr{B}$**.**

Every $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **admits a decomposition** $\mathscr{B} = \mathscr{B}_1 \oplus \mathscr{B}_2,$ **with** $\mathscr{B}_1 \in \mathscr{L}^{\mathtt{w}}$ **controllable and** $\mathscr{B}_2 \in \mathscr{L}^{\mathtt{w}}$ **autonomous. In every such a decomposition,** $\mathscr{B}_1 = \mathscr{B}_{\text{controllable}}$**.**

Controllability and transfer functions

Consider the input/output systems

$$P_1\left(\frac{d}{dt}\right)y_1 = Q_1\left(\frac{d}{dt}\right)u_1, \quad P_2\left(\frac{d}{dt}\right)y_2 = Q_2\left(\frac{d}{dt}\right)u_2,$$

with `determinant`$(P_1) \neq 0$, **and** `determinant`$(P_2) \neq 0$**.**

These two systems have the same transfer function,

$$P_1^{-1}Q_1 = P_2^{-1}Q_2,$$

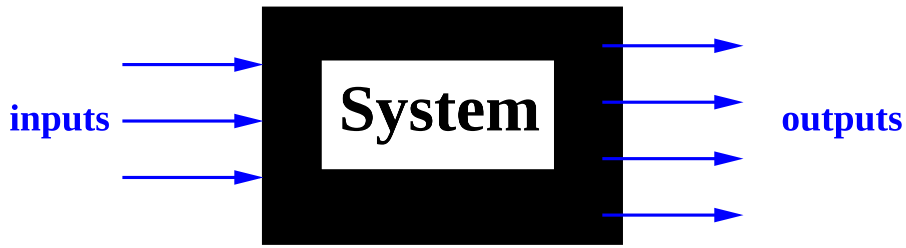
if and only if they have the same controllable part.

Therefore, the transfer function determines only the controllable part of a system.

Rational symbols

Transfer functions

In system theory, it is customary to think of dynamical models in terms of inputs and outputs, viz.



In the LTI case, this leads to transfer functions, $\hat{y} = G(s)\hat{u}$, **with** G **a matrix of rational functions.**

Usually, transfer functions are interpreted in terms of Laplace transforms, with conditions and domains of convergence, and other largely irrelevant mathematical traps.

We now learn to interpret '$y = Gu$**' in terms of differential equations.**

Factorizations of rational matrices

$M \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ **is *left prime* (over** $\mathbb{R}[\xi]$**)** $:\Leftrightarrow$

$$[\![M = FM, \text{ with } F, M \in \mathbb{R}[\xi]^{\bullet\times\bullet}]\!] \Rightarrow [\![F \text{ is unimodular}]\!].$$

It follows from the Smith form that every $M \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ **of full row rank can be written as** $M = FM'$ **with** $F \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ **square and nonsingular, and** $M' \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ **left prime.**

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A ***left coprime*** **polynomial factorization of** $M \in \mathbb{R}(\xi)^{\bullet\times\bullet}$ **is a pair** (P, Q)**, with** $P, Q \in \mathbb{R}[\xi]^{\bullet\times\bullet}, P$ **square and nonsingular,** $M = P^{-1}Q$**, and** $[P \vdots Q]$ **left prime.**

It is easily seen every $M \in \mathbb{R}(\xi)^{\bullet\times\bullet}$ **admits a left coprime polynomial factorization. In the scalar case this simply means writing** M **as a ratio of polynomials without common roots.**

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It follows from the Smith form that every $M \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ of full row rank can be written as $M = FM'$ with $F \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ square and nonsingular, and $M' \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ left prime.

A *left coprime* polynomial factorization of $M \in \mathbb{R}(\xi)^{\bullet\times\bullet}$ is a pair (P,Q), with $P,Q \in \mathbb{R}[\xi]^{\bullet\times\bullet}, P$ square and nonsingular, $M = P^{-1}Q$, and $[P \vdots Q]$ left prime.

It is easily seen every $M \in \mathbb{R}(\xi)^{\bullet\times\bullet}$ admits a left coprime polynomial factorization. In the scalar case this simply means writing M as a ratio of polynomials without common roots.

Right prime and right coprime polynomial factorizations are defined analogously.

ODEs with rational symbols

Defining what a solution is of $R\left(\frac{d}{dt}\right)w = 0$ **poses no difficulties worth mentioning when** R **is a polynomial matrix.**
But ***what do we mean by a solution when*** R ***is a matrix of rational functions?***

ODEs with rational symbols

Let $F \in \mathbb{R}(\xi)^{\bullet\times\bullet}$, and consider the 'differential equation'

$$F\left(\frac{d}{dt}\right)w = 0.$$

$w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **satisfies this differential equation** $:\Leftrightarrow$

$$Q\left(\frac{d}{dt}\right)w = 0$$

where $F = P^{-1}Q$ is a left coprime polynomial factorization.
This definition is independent of the particular left coprime polynomial factorization of F that is taken.

ODEs with rational symbols

By definition, **therefore, the behavior defined by** $F\left(\frac{d}{dt}\right)w = 0$ **is equal to that of** $Q\left(\frac{d}{dt}\right)w = 0$**.** F **is called the** **'symbol'** **associated with this representation.**

ODEs with rational symbols

By definition, **therefore, the behavior defined by** $F\left(\frac{d}{dt}\right)w = 0$ **is equal to that of** $Q\left(\frac{d}{dt}\right)w = 0$**.** F **is called the** **'symbol'** **associated with this representation.**

The use of rational symbols in addition to the polynomial symbols has proven to be very valuable. In Exercise II.6, we see norm-preserving representations. These require rational symbols.

Justification

Assume G **proper. Let** $\frac{d}{dt}x = Ax + Bu, y = Cx + Du$ **be a controllable system with transfer function** G**, i.e.,** $G(\xi) = C(I\xi - A)^{-1}B + D$**. Consider the output nulling inputs**

$$\frac{d}{dt}x = Ax + Bw, 0 = Cx + Dw.$$

These w**'s are exactly those that satisfy** $G\left(\frac{d}{dt}\right)w = 0$**. For** G **not proper, take** $G(\xi) = C(I\xi - A)^{-1}B + D(\xi)$ **with** D **polynomial, and**

$$\frac{d}{dt}x = Ax + Bw, 0 = Cx + D(\frac{d}{dt})w.$$

Justification

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$$\frac{d}{dt}x = Ax + Bw, 0 = Cx + D(\frac{d}{dt})w.$$

Consider the input/output system with transfer function. Take your favorite definition of input/output pairs.

The output nulling inputs are those that satisfy $F\left(\frac{d}{dt}\right)w = 0$**.**

Controllability

Since rational symbols lead to LTIDSs, the notions of controllability, stabilizability, observability, detectability, etc. still pertain to systems defined by rational symbols. In particular, it can be shown that the ‘image representation’

$$w = G\left(\frac{d}{dt}\right)\ell$$

is still controllable with G rational.

Rational transfer functions

Viewing the input/output system

$$y = G\left(\frac{d}{dt}\right)u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix},$$

with G **rational, as a system defined in terms of a rational symbol.**

This leads to a definition of its behavior and of the input/output pairs that is completely independent of Laplace transforms and its mathematical finesses and traps.

In particular, it can be shown that $y = G\left(\frac{d}{dt}\right), w = \begin{bmatrix} u \\ y \end{bmatrix}$

always defines a controllable behavior.

Caveats

$F\left(\frac{d}{dt}\right)$ **is not a map! It associates with an input $u \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ many (a finite-dimensional linear variety) outputs $y \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ such that $y = F\left(\frac{d}{dt}\right)u$.**

$F\left(\frac{d}{dt}\right)$ **is a one-to-many map.**

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$F\left(\frac{d}{dt}\right)$ **is a one-to-many map.**

The operators $F_1\left(\frac{d}{dt}\right)$ **and** $F_2\left(\frac{d}{dt}\right)$ **for** $F_1, F_2 \in \mathbb{R}(\xi)$ **need not commute.**

Recapitulation

Summary

- **There exists a one-to-one relation between the LTIDSs in $\mathscr{L}^{\mathtt{w}}$ and the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{1\times\mathtt{w}}$.**

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- **The variables of a LTIDS allow a componentwise partition in inputs and outputs.**

Summary

- There exists a one-to-one relation between the LTIDSs in $\mathscr{L}^{\mathtt{w}}$ and the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{1\times\mathtt{w}}$.
- The variables of a LTIDS allow a componentwise partition in inputs and outputs.
- There exists tests for verifying controllability and observability of LTIDSs.
- A LTIDS is controllable if and only if it allows an image representation.

Summary

- There exists a one-to-one relation between the LTIDSs in $\mathscr{L}^{\mathtt{w}}$ and the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{1\times\mathtt{w}}$.
- The variables of a LTIDS allow a componentwise partition in inputs and outputs.
- There exists tests for verifying controllability and observability of LTIDSs.
- A LTIDS is controllable if and only if it allows an image representation.
- LTIDSs also allow representations with rational symbols.

End of Lecture II